Birational geometry of moduli spaces of K3 surfaces of low genus

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1 The results

We work over $\mathbb{C}$.

Definition 1.1. A K3 surface of genus g is a K3 surface S (a simply connected projective surface with trivial canonical bundle and ADE singularities at worst) with an ample line bundle L on S such that $c_1(L) \in H^2(S, \mathbb{Z})$ is primitive and $L^2 = 2g - 2$.

Example 1.2. Double covers of plane sextic curves.

Let $\mathcal{F}_g$ be the coarse moduli space of the K3 surfaces of genus g.

Theorem 1.3. *$\mathcal{F}_g$ is a quasi-projective variety of dimension 19, but $\mathcal{F}_g$ is not compact.*

For $g = 2, 3$, there are works of Shah and Laza-O'Grady. We consider the case $g = 4$. Let

- $P_{2,1} \subset \mathcal{F}_4$ be the primitive divisor parametrizing hyperelliptic K3 (S, L), i.e., $\phi_L : S \dashrightarrow \mathbb{P}^4$ is degree 2 map;
- $P_{1,1} \subset \mathcal{F}_4$ the primitive divisor parametrizing unigonal K3, i.e., $\phi_L : S \dashrightarrow \mathbb{P}^4$ maps onto a rational curve;
- $P_{3,1} \subset \mathcal{F}_4$ the primitive divisor parametrizing K3 of genus 4 which is a complete intersection of a singular quadratic and a cubic.

and denote by

$$\mathcal{F}_4^\circ := \mathcal{F}_4 - (P_{1,1} \cup P_{2,1}).$$

We defined the HKL model

$$\mathcal{F}_4(s) := \mathrm{Proj}\,\Big(\bigoplus_{m\geq 0} H^0(\mathcal{F}_4^\circ, m(\lambda - sP_{3,1})|_{\mathcal{F}_4^\circ})\Big),\ s \in [0,1] \cap \mathbb{Q} \tag{1}$$

By the frame work of Looijenga,

$$\mathcal{F}_4(0) = \overline{\mathcal{F}_4^\circ}^{Loo}$$

where $\overline{\mathcal{F}_4^\circ}^{Loo}$ is Looijenga's compactification with respect to divisors $P_{1,1} + P_{2,1}$. We find that

$$\mathcal{F}_4(1) \cong |\mathcal{O}_Q(3)| /\!/ \mathrm{SO}(4).$$

We hope the biratonal map

$$\mathcal{F}_4(0) \dashrightarrow \mathcal{F}_4(1)$$

can be explicitly resolved as we vary $s \in [0, 1] \cap \mathbb{Q}$. So we proposal the following conjecture to predict the birational behaviors of the HKL model 1:

HKL Conjecture 1.4. *Notation as above,*

1. *The section rings* $\bigoplus_{m\geq 0} H^0(\mathcal{F}_4^\circ, m(\lambda - sP_{3,1})|_{\mathcal{F}_4^\circ})$ *are finitely generated for* $s \in [0,1] \cap \mathbb{Q}$*, i.e.,* $\mathcal{F}_4(s)$ *is a projective variety.*

2. *Wall-crossing: Let* s_n *be the n-th value of the set (Wall)*

$$\{\frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{15}, \frac{1}{17}, \frac{1}{19}, \frac{1}{21}, \frac{1}{33}\}.$$

Then for any $s, s' \in (s_n, s_{n+1})$*,* $\mathcal{F}_4(s) \cong \mathcal{F}_4(s')$*. Write*

$$\mathcal{F}_4(s_n, s_{n+1}) = \mathcal{F}_4(s)$$

for any $s \in (s_n, s_{n+1})$*. As s crosses the wall* s_n*, there is a birational map (typically a flip or divisorial contraction) :*

$$\mathcal{F}_4(s_{n-1}, s_n) \xrightarrow{f_n^-} \mathcal{F}_4(s_n) \xleftarrow{f_n^+} \mathcal{F}_4(s_n, s_{n+1})$$

3. *The center of the flip proper transformation of a Shimura subvariety in* $\mathcal{F}_4$*.*

Our main result is

Theorem 1.5 (Greer-Laza-Li-Si-Tian, 2020)**.** *The HKL conjecture 1.4 is true for* $s \in [\frac{1}{3}, 1]$ *or* $s = 0$*.*

2 The motivation and background

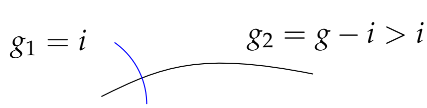
2.1 Motivation 1: Generalization of Hassett-Keel

Recall the Delige-Mumford's moduli spaces

$$\overline{M}_g = \{\text{iso classes of smooth curves of genus g}\} \cup \{\text{nodal}\}$$

compactifies moduli spaces M_g of smooth curves of genus g.

The boundary Δ is a union of divisors Δ_i, $0 \leq g \leq \lfloor \frac{g}{2} \rfloor$, where

$\Delta_i := \{$ 

$\}/\cong$.

Theorem 2.1 (Arbarello-Cornalba, 1987)**.** $\text{Pic}(\overline{M}_g) \otimes \mathbb{Q} = \mathbb{Q}[\Delta_0, \cdots, \Delta_{\lfloor \frac{g}{2} \rfloor}, \lambda]$ *where*

$$\lambda = c_1(\pi_*(\omega_\pi))$$

is the Hodge line bundle for universal family $\mathcal{C} \to \overline{M}_g$*.*

- 2005, Hassett study log canonical models for $\overline{M}_2$.

- 2009, 2011, after [BCHM], Hassett-Hyeon study the log canonical models

$$\overline{M}_g(\beta) := \mathrm{Proj}\,(\bigoplus_{m\geq 0} H^0(\overline{M}_g, m(K_{\overline{M}_g} + \beta\Delta_0 + \frac{1+\beta}{2}\Delta_1 + \beta\Delta_2 + \cdots + \beta\Delta_{\lfloor \frac{g}{2} \rfloor})),\ \ \beta \in [0,1] \cap \mathbb{Q}$$

$\overline{M}_g(\beta)$ compares various compactifications of M_g, for example,

1. By the classical work of Harris etc,

$$\overline{M}_g(\beta) \cong \overline{M}_g,\ \text{ for } \frac{9}{11} < g \leq 1$$

2. Hassett-Hyeon showed that

$$\overline{M}_g(\beta) \cong \overline{M}_g^{ps},\ \text{ for } \frac{7}{10} < g \leq \frac{9}{11}$$

   where the space $\overline{M}_g^{ps}$ is the coarse moduli space of curves of genus g with cusps at worst and without elliptic tails.

Moreover, they find as the coefficient β crosses $\frac{9}{11}$, the birational morphism of these two coarse moduli space is a divisorial contraction

$$\overline{M}_g(\frac{9}{11}, 1] \longrightarrow \overline{M}_g(\frac{7}{10}, \frac{9}{11}],$$

which contracts Δ_1 to the loci of curves with a cusp.

2.2 Motivation 2: Comparing various compactification

1. Hodge theoretical side (Arithmetic compactifications)

   The primitive part of second cohomology group $H^2(S,\mathbb{Z})_{\mathrm{prim}} := c_1(L)^{\perp}$ is isometric to the even lattice

$$\Lambda_g = \mathbb{Z}\omega \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2},$$

   where $\langle \omega, \omega \rangle = -(2g-2)$. The global Torelli theorem implies the period map

$$p : \mathcal{F}_g \to \mathcal{D}^g / \Gamma_g,$$

   is isomorphic where

$$\mathcal{D}^g := \{z \in \mathbb{P}(\Lambda_g \otimes \mathbb{C}) :\ \langle z, z \rangle = 0,\ \langle z, \overline{z} \rangle > 0\}^+$$

   is the period domain and Γ_g is the monodromy group.

   As a locally symmetric variety, $\mathcal{F}_g$ admits many compactifications from arithemtic side, for example,

   - 1966, Baily-Borel: construct compactification $\mathcal{F}_g^*$.

Theorem 2.2 (Baily-Borel 1966). *The graded ring $\bigoplus_{k\geq 0} M_k(\Gamma)$ is a finitely generated $\mathbb{C}$-algebra and there is isomorphism*

$$\mathcal{F}_g^* \cong \mathrm{Proj}\,(\bigoplus_{k\geq 0} M_k(\Gamma)) \tag{2}$$

where $M_k(\Gamma)$ is the space of weight k modular forms with level Γ

$$M_k(\Gamma) := \{\, f : \mathcal{D}_g \to \mathbb{C} \mid f \text{ holomorphic}, \ f(\lambda z) = \lambda^k f(z),\ \lambda \in \mathbb{C}^* \\ f(\gamma \cdot z) = f(z),\ \gamma \in \Gamma \,\}.$$

- 1975, Ash-Mumford-Rappoet-Tai, Toroidal compactifcation.
- 2003, Looijenga: Inspired by Baily-Borel, develop a compactification for a complement of hyperplane arrangement.

$$\begin{array}{ccc} \widetilde{\mathcal{F}}_4^{\mathcal{H}} & \xrightarrow{\pi_2} & \mathcal{F}_4^{\Sigma(\mathcal{H})} \\ {\scriptstyle \pi_1}\downarrow & \overset{\widetilde{\pi}}{\searrow} & \downarrow{\scriptstyle \pi_{\mathcal{H}}} \\ \overline{\mathcal{F}}_4^{\mathcal{H}} & \dashrightarrow & \mathcal{F}_4^* \end{array}$$

2. Projective model side (GIT compactifications): for K3 surfaces (S, L) of low genus, its projective model, i.e., the rational map

$$\phi_L : \ S \dashrightarrow \mathbb{P}^g = \mathbb{P}H^0(S, L)^\vee \tag{3}$$

can be classified due to the work of Saint Donat , Mukai etc. For example,

(a) $g = 2$ ϕ_L is either a double cover of $\mathbb{P}^2$ branched along a $C \in |\mathcal{O}(6|$ or onto a conic in $\mathbb{P}^2$.

(b) $g = 4$, if $[S, L] \in \mathcal{F}_4 - (P_{1,1} \cup P_{2,1})$, then ϕ_L is birational onto a $(2, 3)$ complete intersections in $\mathbb{P}^3$.

Once the projective model is explicit given, we can construct their Hilbert scheme (or chow variety) as parameter space explicitly. Then Mumford's GIT provides a natural compactification.

For $g = 4$, we can construct VGIT: $\mathcal{Q} \subseteq \mathbb{P}^4 \times |\mathcal{O}_{\mathbb{P}^4}(2)|$ be the universal quadratic. Define vector bundle

$$E := (p_2)_* p_1^* \mathcal{O}(2)$$

where $p_1 : \mathbb{P}^4 \times |\mathcal{O}_{\mathbb{P}^4}(2)| \to \mathbb{P}^4$ and $p_1 : \mathbb{P}^4 \times |\mathcal{O}_{\mathbb{P}^4}(2)| \to |\mathcal{O}_{\mathbb{P}^4}(2)|$ are two projections. Note that that E is sitting in the following exact sequence,

$$0 \to H^0(\mathbb{P}^4, \mathcal{O}(1)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^{14}}(-1) \to H^0(\mathbb{P}^4, \mathcal{O}(3)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^{14}} \to E \to 0$$

The associated projective bundle

$$\pi : \mathbb{P}(E) \to |\mathcal{O}_{\mathbb{P}^4}(2)| \cong \mathbb{P}^{14}$$

parametrizes all $(2, 3)$-schemes in $\mathbb{P}^n$, whose fiber at $[q]$ is

$$E_{[q]} = \{f \in |\mathcal{O}_{\mathbb{P}^4}(3)| : q \text{ is not a factor of } f\}.$$

The Picard group of $\mathbb{P}(E)$ is spanned by $\eta = \pi^* \mathcal{O}_{\mathbb{P}^{14}}(1)$ and $h = \mathcal{O}_{\mathbb{P}(E)}(1)$. Set

$$H_t = th + \eta,$$

then H_t is ample if and only if $0 < t < \frac{1}{2}$.

We define the VGIT model $\overline{\mathfrak{M}}(t)$ as

$$\overline{\mathfrak{M}}(t) = \mathbb{P}(E) /\!/_t \mathrm{SL}(5) := \mathrm{Proj} R(\mathbb{P}(E), H_t)^{\mathrm{SL}(5)}$$

with respect to the line bundle H_t.

3. K-stability or KSBA side (Birtional geometry): just mention ADL's work on compactifying plane setics using K-stability.

4. Others $\cdots$.

3 Proof

Our proof based on VGIT theory and NL-number computations.